On supersymmetries in nonrelativistic quantum mechanics

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Abstract

One-dimensional nonrelativistic systems are studied when time-independent potential interactions are involved. Their supersymmetries are determined and their closed subsets generating kinematical invariance Lie superalgebras are pointed out. The study of even supersymmetries is particularly enlightened through the already known symmetries of the corresponding Schrödinger equation. Three tables collect the even, odd, and total supersymmetries as well as the invariance (super)algebras.

1 Introduction

In the seventies, systematic studies of *symmetries* in nonrelativistic quantum mechanics (NRQM) had already been realized when the Schrödinger equation involves time-independent potential interactions [1, 2, 3, 4]. Some of them have considered one-dimensional [2, 4], three-dimensional [1], or n-dimensional [3] (space) systems.

Recently [5] we have added new information by visiting higher-order symmetries of such one-dimensional Schrodinger equations and by considering the third-order context as an effective case, (Remember that the usual well-known symmetries of the Schrödinger equation correspond to first- and second-order operators.)

Since the event of supersymmetry in theoretical particle physics [6] and its implications [7] in supersymmetric (nonrelativistic) quantum mechanics (SSQM), we are dealing with supersymmetric wave equations, including time-independent superpotentials [hereafter denoted $W(x)$ in one-dimensional problems]. Such supersymmetric equations are usually put in the following matrix form:

$$i\partial_t\chi(x,t) = \left[-\frac{1}{2}\partial_x^2 + \frac{1}{2}W^2(x) + \frac{1}{2}W'(x)\sigma_3\right]\chi(x,t), \tag{1.1}$$

where $\partial_t = \partial/\partial t$, the prime refers to the space derivative $\partial_x \equiv \partial/\partial x$, σ_3 is the Pauli 2×2 matrix diag$(1,-1)$, and correlatively $\chi(x,t)$ is a two-component wave function. For simplicity we consider systems with unit masses and choose $\hbar = 1$ in this quantum context.

In Hamiltonian form, Eq. (1.1) contains as expected, the bosonic part H_b and the fermionic part H_f given by

$$H_b = -\frac{1}{2}\partial_x^2 + \frac{1}{2}W^2(x), \quad H_f = \frac{1}{2}W'(x)\sigma_3, \quad H_{SS} = H_b + H_f, \tag{1.2}$$

showing that superpotentials $W(x)$ are simply related to usual potentials $U(x)$ by

$$U(x) = \frac{1}{2} W^2(x). \tag{1.3}$$

Such supersymmetric equations (1.1) and characteristics (1.2) have already suggested new methods and results [8, 9] by considering even and odd symmetries in connection with the so-called Lie extended method developed in quantum physics [10].

As a particular context that has been recently visited [8], let us mention the case of the one-dimensional supersymmetric harmonic oscillator corresponding to

$$W(x) = \omega x \;\Leftrightarrow\; U(x) = \frac{1}{2}\omega^2 x^2. \tag{1.4}$$

Here we want to study systematically the supersymmetries as well as their superstructures subtended by the SSQM-Schrödinger equation (1.1) for arbitrary superpotentials $W(x)$. Such a program will appear in the following sections where we will get a complete classification of all solvable (admissible) interactions admitting nontrivial supersymmetries.

The contents of this paper are distributed as follows. In Section 2 we come back on the Boyer results [3], but for one-dimensional systems. We present the corresponding study in a particularly convenient way for our purpose in the supersymmetric context developed in Section 3. There we first determine the even supersymmetries (Section 3.1), then second the odd ones (Section 3.2), and finally we superpose both contexts (Section 3.3) and get only four priviledged classes of superpotentials. All the results are quoted in three respective tables, where we mention the largest invariance Lie (super)algebras generated by the corresponding operators. Remember that these extended (super)symmetries do not necessarily close. Section 4 finally contains some comments on isomorphic structures and some conclusions.

2 Going back on symmetries in NRQM

In fact, Boyer [3] has already solved the problem of determining all the symmetries of the Schrödinger equation in n dimensions. Applied to the simplest one-dimensional context, this problem consists in the resolution of the following invariance condition:

$$[\Delta, Q] = i\lambda\Delta, \quad \lambda \equiv \lambda(x,t). \tag{2.1}$$

Here the Schrödinger equation defines the Δ operator on the form

$$\Delta\psi(x,t) \equiv \left(-i\partial_t - \frac{1}{2}\partial_x^2 + U(x)\right)\psi(x,t), \tag{2.2}$$

while we are searching for symmetry operators Q of (at most) second order with respect to space derivatives, i.e.,

$$Q \equiv ia(x,t)\partial_t + ib(x,t)\partial_x + ic(x,t), \tag{2.3}$$

a, b, and c being arbitrary functions. Let us mention that higher-order symmetry operators have recently been considered elsewhere in the same context [5].

The above problem leads to a set of partial differential equations giving rise to x-independent functions a_0, b_0, and c_0 and to the following first-order partial differential equation on $U(x)$:

$$\left(\frac{1}{2}\dot{a}_0 x + b_0\right) U'(x) + \dot{a}_0 U(x) = -\frac{1}{4}\dddot{a}_0 x^2 - \ddot{b}_0 x - i\dot{c}_0 + \frac{i}{4}\ddot{a}_0, \tag{2.4}$$

where the overdots evidently refer to time derivatives. The general solution of Eq. (2.4) appears as the sum of the general solution $U_0(x)$ of the homogeneous equation and of a particular solution $U_1(x)$ of the inhomogeneous equation. Due to the at most quadratic dependence on x in the inhomogeneous part of Eq. (2.4), Boyer has proposed to search for the general solution [3]

$$U(x) = U_0(x) + U_1(x), \tag{2.5}$$

with

$$U_1(x) = \frac{1}{2}\alpha x^2 + \beta x + \gamma, \quad \alpha, \beta, \gamma = \text{arbitrary constants}, \tag{2.6}$$

and to distinguish three cases according to $\alpha = 0$, $\alpha = \omega^2 > 0$, or $\alpha = -\omega^2 < 0$.

If the whole discussion takes place in Boyer's work [3], let us notice here the following results and properties in order to exploit them in the supersymmetric context.

In the free case, i.e., $U = 0$ ($U_0 = 0 = U_1$), as already obtained by Niederer [1], we get at most six symmetries whose commutation relations lead to the largest invariance algebra seen as a semidirect sum ($\square$) of the so-called "conformal" algebra so$(2,1)$ and the Heisenberg algebra $h(2)$.

In the interacting case but when $U_0 = 0$, we want to point out a new result, i.e., the potential

$$U(x) = U_1(x) \tag{2.7}$$

leads to isomorphic structures so$(2,1)\square h(2)$ a s in the free case ($\alpha = \beta = \gamma = 0$) whatever are the constants α, β, and γ. As particular cases, the harmonic oscillator corresponds to $U_0 = 0$, $\alpha = \omega^2$, $\beta = \gamma = 0$ [see Eqs. (1.3) and (2.7)] while the linear potential to $U_0 = 0$, $\alpha = 0$, i.e.,

$$U(x) = \beta x + \gamma. \tag{2.8}$$

Consequently, these cases also admit six symmetries and there exist changes of variables connecting all the equations including the potential forms (2.7) for arbitrary α, β and γ, with the free Schrödinger equation. Let us just quote as an example some formulas corresponding to the change of variables between the case $\alpha = \omega^2 > 0$ and the free case ($\alpha = \beta = \gamma = 0$). It can be shown that the relations

$$t_1 = (1/\omega)\tan^{-1}\omega t_2, \quad x_1 = (1 + \omega^2 t_2^2)^{-1/2}(x + \beta/\omega^2) - \beta/\omega^2 \tag{2.9}$$

correspond to a well-defined change of variables between the above interacting case (indices 1) and the free case (indices 2) implying a modification in the corresponding Schrödinger wave functions according to

$$\Psi_1(x_1,t_1) = (1+\omega^2 t_2^2)^{1/4} \exp\left[\left(\frac{i\beta^2}{2\omega^3} - \frac{i\gamma}{\omega}\right)\tan^{-1}\omega t_2 - \frac{i\beta^2}{2\omega^3}\frac{t_2}{1+\omega^2 t_2^2}\right] \times \exp\left[-\frac{it_2 x}{2(1+\omega^2 t_2^2)}(2\beta+\omega^2 x)\right]\Psi_2(x_2,t_2). \tag{2.10}$$

As a more particular case included in this comment, i.e., the harmonic oscillator case with $\alpha = \omega^2$, $\beta = \gamma = 0$, we immediately recover the Niederer result [1]:

$$\Psi_1(x_1,t_1) = (1+\omega^2 t_2^2)^{1/4} \exp\left[-\frac{it_2\omega^2 x^2}{2(1+\omega^2 t_2^2)}\right]\Psi_2(x_2,t_2). \tag{2.11}$$

When nontrivial potentials $U_0(x) \neq 0$ are considered, only two cases have to be mentioned: either U_0 is arbitrary, then all the potentials

$$U(x) = U_0 + \frac{1}{2}\alpha x^2 + \beta x + \gamma \tag{2.12}$$

lead to at least two symmetries (the corresponding Hamiltonian and the identity operator); or U_0 takes the nonzero form

$$U_0(x) = \delta/(\mu x+\epsilon)^2, \quad \mu \neq 0, \tag{2.13}$$

then all the potentials

$$U(x) = \delta/(\mu x+\epsilon)^2 + \frac{1}{2}\alpha x^2 + \beta x + \gamma, \quad [(\mu,\epsilon) = (\alpha,\beta) \text{ if } \alpha \neq 0], \tag{2.14}$$

lead to four symmetries generating the direct sum so$(2,1) \oplus I$.

Such properties are of special interest for the following discussion of the supersymmetries of Eq. (1.1) in particular.

3 Going to supersymmetries in SSQM

Already considered [8, 9] for only one-dimensional supersymmetric harmonic oscillators characterized by interacting terms given in Eq. (1.4), the search for the largest number(s) of one-parameter Lie algebras can be extended to arbitrary (one-dimensional) systems. This asks for considering the problem (2.1) but with an operator ASS defined by the supersymmetric equation (1.1) and with symmetry operators Q containing even ($\vec{0}$) and odd (1) parts [8] according to

$$Q = Q_0 + Q_1, \tag{3.1}$$

refering to the expected graduation in the supercontext. Consequently, let us decompose our program in three steps: first to study the even supersymmetries (Section 3.1) by

exploiting the results contained in Section 2 on symmetries; second to develop the new odd context and to get the corresponding supersymmetries (Section 3.2); and, third, to superpose the two sets of results (Section 3.3) in order to obtain the complete classification of solvable interactions in SSQM.

Let us just notice here that, in connection with Eq. (1.1), we understand that even and odd considerations are directly connected with 2×2 Pauli matrices or more correctly with the Clifford algebra $Cl_2 \equiv \{\sigma_0 \equiv I_2, \sigma_1, \sigma_2, \sigma_3\}$, where we easily distinguish the even matrices (σ_0, σ_3) and the odd ones (σ_1, σ_2) the usual fundamental representation [11]. Such a remark directly enlightens Sections 3.1 and 3.2.

3.1 Even supersymmetries in SSQM

The invariance condition (2.1) is replaced here by

$$[\Delta_{SS}, Q_0] = i\lambda_0 \Delta_{SS}, \quad \lambda_0 = \lambda_0(x,t), \tag{3.2}$$

where

$$\Delta_{SS} = \begin{pmatrix} -i\partial_t - \frac{1}{2}\partial_x^2 + \frac{1}{2}W^2(x) + \frac{1}{2}W'(x) & 0 \\ 0 & -i\partial_t - \frac{1}{2}\partial_x^2 + \frac{1}{2}W^2(x) - \frac{1}{2}W'(x) \end{pmatrix} \tag{3.3}$$

and

$$Q_0 = \begin{pmatrix} \begin{matrix} -i(a_0+a_3)\partial_t + i(b_0+b_3)\partial_x \\ + i(c_0+c_3) \end{matrix} & 0 \\ 0 & \begin{matrix} i(a_0-a_3)\partial_t + i(b_0-b_3)\partial_x \\ + i(c_0-c_3) \end{matrix} \end{pmatrix} \tag{3.4}$$

due to the explicit forms of the even matrices σ_0 and σ_3. Such a problem is equivalent to a set of two distinct ones in NRQM, as discussed in Section 2. Indeed we get the two invariance conditions

$$\left[-i\partial_t - \frac{1}{2}\partial_x^2 + V_1(x), i(A_1\partial_t + B_1\partial_x + C_1)\right] = i\Lambda_1\left(-i\partial_t - \frac{1}{2}\partial_x^2 + V_1(x)\right) \tag{3.5a}$$

and

$$\left[-i\partial_t - \frac{1}{2}\partial_x^2 + V_2(x), i(A_2\partial_t + B_2\partial_x + C_2)\right] = i\Lambda_2\left(-i\partial_t - \frac{1}{2}\partial_x^2 + V_2(x)\right), \tag{3.5b}$$

where the arbitrary functions A_i, B_i, C_i and Λ_i $(i = 1, 2)$ are evidently simply related to the above $a_0, a_3, \ldots$, while we have defined

$$V_1(x) = \frac{1}{2}W^2(x) + \frac{1}{2}W'(x) \tag{3.6a}$$

and

$$V_2(x) = \frac{1}{2}W^2(x) - \frac{1}{2}W'(x), \tag{3.6b}$$

these functions being nothing else than the potentials associated with superpartners [12].

Table 1.

Classes	$\mathcal{E}$ Supersymmetries and their Lie algebras	Number of $\mathcal{E}$ supersymmetries	Explicit forms of associated superpotentials	Characteristics
1	$[\mathrm{so}(2,1)\square h(2)] \oplus [\mathrm{so}(2,1)\square h(2)]$	12	$W(x) = ax + b$	Free case Linear case Harmonic oscillator [9]
2	$[\mathrm{so}(2,1)\square h(2)] \oplus [\mathrm{so}(2,1)\square \mathrm{gl}(1)]$	10	$W(x) = \pm 1/(x+c)$	Coulomb ($c = 0$)
3			$W(x) = ax \pm 1/x$	Calogero ($a = \omega$)
4	$[\mathrm{so}(2,1) \oplus \mathrm{gl}(1)] \oplus [\mathrm{so}(2,1) \oplus \mathrm{gl}(1)]$	8	$W(x) = ax + c/x$	$a \neq 0,\ c \neq \pm 1$
5			$W(x) = (fx+g)/(cx^2+dx+h)$	$f \neq 0,\ c \neq 0,$ $f \neq \pm c,\ c \neq 0$
6	$[\mathrm{so}(2,1)\square h(2)] \oplus [\mathrm{so}(2) \oplus \mathrm{gl}(1)]$	8	$W(x) = ax + b + \frac{ce^{\mp ax^2 \mp 2bx}}{d + cfe^{\mp ax^2 \mp 2bx} dx}$	$(a,b) \neq (0,0)$ $d \neq 0,\ c \neq 0$
7			$W(x) = \frac{c}{dx+f} + \frac{g}{h(dx+f)^{\pm 2c/d} \pm [g/(d \mp 2c)](dx+f)}$	$d \neq \pm 2c$ $g \neq 0,\ c \neq 0$
8	$[\mathrm{so}(2,1) \oplus \mathrm{gl}(1)] \oplus [\mathrm{so}(2) \oplus \mathrm{gl}(1)]$	6	$W(x) = \frac{c}{\pm 2cx+d} + \frac{f}{(\pm 2cx+d)(g+(f/2c)\ln\lvert \pm 2cx+d \rvert)}$	$f \neq 0,\ a \neq 0,$ $d \neq 0$
9			$W(x) = ax + b + \frac{c}{dx+f} + \frac{g(dx+f)^{\mp 2c/d} e^{\mp ax^2 \mp 2bx}}{h \pm gfe^{\mp ax^2 \mp 2bx}(dx+f)^{\mp 2c/d} dx}$	$c \neq 0, \pm 1$
10	$[\mathrm{so}(2) \oplus \mathrm{gl}(1)] \oplus [\mathrm{so}(2) \oplus \mathrm{gl}(1)]$	4	$W(x) \neq$ above forms	$\cdots$

By combining and superposing the potentials (2.6), (2.7), (2.12), and (2.14) in known NRQM, we immediately deduce the cases leading to (at most) 12 even supersymmetries or to 10, 8, 6, and (at least) 4 of them. The (six) associated invariance Lie algebras are also easily determined (see Table 1): the largest one is seen as the direct sum $[\mathrm{so}(2,1)\square h(2)] \oplus [\mathrm{so}(2,1)\square h(2)]$ and the smallest one as $[\mathrm{so}(2)\oplus\mathrm{gl}(1)]\oplus[\mathrm{so}(2)\oplus\mathrm{gl}(l)]$. Such direct sums evidently come from the superposition of both projections by $P_\pm = \frac{1}{2}(\sigma_0 \pm \sigma_3)$ of the ordinary symmetries obtained in NRQM. Moreover, by exploiting the explicit forms obtained for V_1, and V_2 in NRQM, it is possible through Eqs. (3.6) to get the corresponding superpotentials $W(x)$ entering in Eq. (1.1). Let us, for example, treat the case leading to the maximal number of even supersymmetries, i.e., let us consider

$$V_1(x) = \frac{1}{2}\alpha_1 x^2 + \beta_1 x + \gamma_1 \tag{3.7}$$

and

$$V_2(x) = \frac{1}{2}\alpha_2 x^2 + \beta_2 x + \gamma_2.$$

Through Eqs. (3.6) we readily get the linear superpotential

$$W(x) = ax + b, \quad a, b = \text{const}, \tag{3.8}$$

which also corresponds to the free case ($a = b = 0$) as well as to the one-dimensional harmonic oscillator ($a = \omega$, $b = 0$). Here are associated 12 even Supersymmetries, an old result quoted in the even notations [9],

$$[(H_B, C_\pm, I, P_\pm)\sigma_0] \quad \text{and} \quad [(H_B, C_\pm, I, P_\pm)\sigma_3]. \tag{3.9}$$

The general discussion leads to specific families of supcrpotentials given in Table 1 simultaneously with their associated invariance Lie algebras and their dimensions. Let us point out the Coulomb-like and Calogero-like forms admitting ten even supersymmetries.

3.2 Odd supersymmetries in SSQM

With the operator $\Delta_{SS} \equiv$ (3.3) we have now to exploit the invariance condition

$$[\Delta_{SS}, Q_1] = i\lambda_1 \Delta_{SS}, \quad \lambda_1 = \lambda_1(x,t), \tag{3.10}$$

where

$$Q_1 = \begin{pmatrix} 0 & \begin{matrix}(ia_1+a_2)\partial_t + (ib_1+b_2)\partial_x \\ + ic_1 + c_2\end{matrix} \\ \begin{matrix}(ia_1-a_2)\partial_t + (ib_1-b_2)\partial_x \\ + ic_1 - c_2\end{matrix} & 0 \end{pmatrix} \tag{3.11}$$

due to the explicit forms of the odd matrices σ_1 and σ_2. Such a problem leads to a set of two third-order equations, which take the following forms, where the functions α_i, β_i, and γ_i, are the x-independent parts of a_i, b_i, and c_i $(i = 1, 2)$, respectively:

$$\begin{aligned} &\frac{i}{8}\alpha_2 W''' - \frac{3i}{4}\alpha_2 W^2 W' - \frac{i}{4}\ddot{\alpha}_2 x^2 W' - i\dot{\beta}_2 x W' - \frac{1}{4}\dot{\alpha}_2 W' + \gamma_2 W' \\ &\qquad - \frac{i}{4}\dot{\alpha}_1 W^2 - \frac{i}{2}\ddot{\alpha}_2 x W - i\dot{\beta}_2 W - \frac{i}{4}\dddot{\alpha}_1 x^2 - i\ddot{\beta}_1 x - \frac{1}{4}\ddot{\alpha}_1 + \dot{\gamma}_1 = 0 \end{aligned} \tag{3.12a}$$

and

$$\begin{aligned} &-\frac{i}{8}\alpha_1 W''' + \frac{3i}{4}\alpha_1 W^2 W' + \frac{i}{4}\ddot{\alpha}_1 x^2 W' + i\dot{\beta}_1 x W' + \frac{1}{4}\dot{\alpha}_1 W' - \gamma_1 W' \\ &\qquad - \frac{i}{4}\dot{\alpha}_2 W^2 + \frac{i}{2}\ddot{\alpha}_1 x W + i\dot{\beta}_1 W - \frac{i}{4}\dddot{\alpha}_2 x^2 - i\ddot{\beta}_2 x - \frac{1}{4}\ddot{\alpha}_2 + \dot{\gamma}_2 = 0. \end{aligned} \tag{3.12b}$$

These inhomogeneous nonlinear equations simultaneously admit the particular solution (3.8), so that we can in this supersymmetric context search for the general solution

$$W(x) = W_0(x) + W_1(x), \tag{3.13}$$

where we accept

$$W_1(x) = ax + b, \quad a, b = \text{const}, \tag{3.14}$$

and where we have to discuss the two cases $W_0 = 0$ or $W_0 \neq 0$. Such a discussion is analogous to the one developed in NRQM through Eqs. (2.5), (2.7), and (2.12).

Let us (first) consider $W_0(x) = 0$, so that

$$W(x) = W_1(x) = ax + b. \tag{3.15}$$

Here again we can distinguish $a = 0$ and $a \neq 0$. Both cases lead to 12 (odd) supersymmetries. In particular, if $a = \omega$, $b = 0$, this context coincides with the supersymmetric

harmonic oscillator already visited and characterized by this maximal number of odd supersymmetries [9].

Then let us (second) take $W_0 \neq 0$ and insert the value (3.13) in Eqs. (3.12). For arbitrary W_0, it is once again interesting to distinguish between the two cases $a = 0$ and $a \neq 0$. Such a discussion leads to at least two odd super-symmetries, which appear on the forms

$$Q_1^{(1)} = i\sigma_1\partial_x - \sigma_2 W(x) \quad \text{and} \quad Q_1^{(2)} = i\sigma_2\partial_x + \sigma_1 W(x). \tag{3.16}$$

We also point out that they generate the simplest $N = 2$ superalgebra sqm(2) initially introduced by Witten [7]. We effectively have

$$\begin{aligned} &\{Q_1^{(1)}, Q_1^{(1)}\} = \{Q_1^{(2)}, Q_1^{(2)}\} = -2\partial_x^2 + 2W^2(x) + 2\sigma_3 W'(x) = 4H_{SS} = 4i\partial_t, \\ &\{Q_1^{(1)}, Q_1^{(2)}\} = 0, \quad [Q_1^{(i)} H_{SS}] = 0, \quad i = 1, 2. \end{aligned} \tag{3.17}$$

In order to complete our classification of admissible interactions characterized by odd supersymmetries, we now want to determine the impact due to each symmetry on the superpotentials $W_0(x)$ left arbitrary in Eq. (3.13). This can be studied by quoting the general forms of our arbitrary functions α_i, β_i, and γ_i (depending only on t) appearing in Eq. (3.12). When $a = 0$, these functions are obtained in terms of the 12 arbitrary constants $A_{(0)}$, $B_{(0)}$, $C_{(0)}$, $D_{(0)}$, $E_{(0)}$, $F_{(0)}$, $G_{(0)}$, $H_{(0)}$, $K_{(0)}$, $L_{(0)}$, $M_{(0)}$, and $N_{(0)}$ (the subscript refers to this $a = 0$ case) as follows:

$$\begin{aligned} \alpha_1(t) &= \frac{1}{2}A_{(0)}t^2 + B_{(0)}t + C_{(0)}, \\ \alpha_2(t) &= \frac{1}{2}D_{(0)}t^2 + E_{(0)}t + F_{(0)}, \\ \beta_1(t) &= -\frac{1}{4}bD_{(0)}t^2 + G_{(0)}t + H_{(0)}, \\ \beta_2(t) &= \frac{1}{4}bA_{(0)}t^2 + K_{(0)}t + L_{(0)}, \\ \gamma_1(t) &= \frac{1}{4}A_{(0)}t\left(1 + \frac{3i}{2}b^2t\right) + \frac{i}{4}B_{(0)}b^2t + iK_{(0)}bt + M_{(0)}, \\ \gamma_2(t) &= \frac{1}{4}D_{(0)}t\left(1 + \frac{3i}{2}b^2t\right) + \frac{i}{4}E_{(0)}b^2t - iG_{(0)}bt + N_{(0)}, \end{aligned} \tag{3.18}$$

while, when $a \neq 0$, they are given by

$$\begin{aligned} \alpha_1(t) &= A\exp(iat) + B\exp(-iat) + iC\exp(3iat) - iD\exp(-3iat), \\ \alpha_2(t) &= E\exp(iat) + F\exp(-iat) + C\exp(3iat) + D\exp(-3iat), \\ \beta_1(t) &= \frac{i}{2}b[A\exp(iat) - B\exp(-iat)] + i[K\exp(2iat) - L\exp(-2iat)] \\ &\quad - \frac{3}{2}b[C\exp(3iat) - D\exp(-3iat)] + M, \\ \beta_2(t) &= \frac{1}{2}b[E\exp(iat) - F\exp(-iat)] + K\exp(2iat) + L\exp(-2iat) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{3i}{2} b[C\exp(3iat) - D\exp(-3iat)] + G,\\
\gamma_1(t) &= \exp(iat)\left(iP + \frac{1}{4}aE + \frac{1}{4}b^2E + \frac{i}{4}aA + \frac{i}{4}b^2A\right)\\
&- \exp(-iat)\left(iQ - \frac{1}{4}aF + \frac{1}{4}b^2F + \frac{i}{4}aB - \frac{i}{4}b^2B\right)\\
&+ 2ib[K\exp(2iat) + L\exp(-2iat)] - C\exp(3iat)\left(\frac{3}{4}a + \frac{9}{4}b^2\right)\\
&- D\exp(-3iat)\left(\frac{3}{4}a - \frac{9}{4}b^2\right),\\
\gamma_2(t) &= P\exp(iat) + Q\exp(-iat) + 2b[K\exp(2iat) - L\exp(-2iat)]\\
&+ \frac{3i}{4}C\exp(3iat)(3b^2 + a) + \frac{3i}{4}D\exp(-3iat)(3b^2 - a), \qquad (3.19)
\end{aligned}
$$

showing once again 12 arbitrary constants.

As an example, let us introduce (3.13) and (3.14) in Eqs. (3.12) when $a = 0$, i.e., when $W(x) = W_0(x) + b$. By exploiting the relations (3.18) when only the constant $A_{(0)}$ is nonzero, we finally obtain three time-independent conditions on $W_0(x)$, which are

$$
\begin{aligned}
&W_0''' - 6W_0(W_0 + 2b)W_0' = 0,\\
&2bxW_0' + W_0(W_0 + 4b) = 0,\\
&x(xW_0' + 2W_0) = 0.
\end{aligned}
$$

They are easily handled for getting the unique solution $W_0(x) = 0$. This $A_{(0)}$ context leads to the symmetry operator

$$
\begin{aligned}
Q_1^{A(0)} &= \sigma_1\left(\frac{i}{2}t^2\partial_t + \frac{i}{2}tx\partial_x + \frac{1}{4}x + \frac{1}{4}bt^2W - \frac{1}{8}t^2W^2 + \frac{i}{4}t - \frac{3}{8}b^2t^2\right)\\
&+ \sigma_2\left(\frac{i}{4}bt^2\partial_x + \frac{1}{2}btx + \frac{i}{8}t^2W' - \frac{i}{4}t^2W\partial_x - \frac{1}{2}txW\right). \qquad (3.20)
\end{aligned}
$$

By collecting all the similar information for the whole set of odd (super)symmetries associated with the case $a = 0$, we finally determine besides the maximal (12) and minimal (2) numbers of odd supersymmetries already obtained that only three intermediate cases can occur: either $W(x) = \pm 1/x$ admits ten (odd) symmetries; or $W(x) = \pm c/x$ admits four (odd) symmetries if $c \neq \pm 1$; or $W(x) = b + W_0(x)$ admits four (odd) symmetries if we take account of an extradependence in terms of Legendre functions.

When $a \neq 0$, the developments are more elaborate but the complete results can once again be obtained through the resolution of relatively complicated nonlinear differential equations. Just as an example with

$$
W(x) = W_0(x) + ax + b, \quad a \neq 0, \qquad (3.21)
$$

let us use Eqs. (3.12), (3.13), (3.14) and insert Eq. (3.19) when only the arbitrary constant C is nonzero. This case is then characterized by the functions

$$
\alpha_1(t) = iC\exp(3iat), \quad \alpha_2(t) = C\exp(3iat),
$$

$$\begin{aligned}
\beta_1(t) &= -\frac{3}{2}bC\exp(3iat), \quad \beta_2(t) = \frac{3i}{2}bC\exp(3iat), \\
\gamma_1(t) &= -\frac{3}{4}aC\exp(3iat) - \frac{9}{4}b^2C\exp(3iat), \\
\gamma_2(t) &= \frac{9i}{4}b^2C\exp(3iat) + \frac{3i}{4}aC\exp(3iat).
\end{aligned} \tag{3.22}$$

Thus we get the following third-order (nonlinear) differential equation on $W_0(x)$:

$$W_0^{'''} = 6W_0^2W_0' - 12(ax+b)^2W_0' + 12(ax+b)W_0W_0' - 36a(ax+b)W_0, \tag{3.23}$$

or, in terms of $W(x) \equiv (3.21)$,

$$W^{'''} = 6W^2W' - 18(ax+b)^2W' - 6aW^2 - 36a(ax+b)W + 54a(ax+b^2). \tag{3.24}$$

Such an equation has already been quoted and solved in the literature [13]: it corresponds to the derivative of a Painlevé IV-type equation and leads to the solution [13]

$$W(x) = \epsilon\frac{dP_4}{dx}(0,\delta;x) + \frac{\epsilon}{\mu}\left(x + \frac{b}{a}\right). \tag{3.25}$$

Now, by collecting all the corresponding information for the whole set of odd (super) symmetries associated with the case $a \neq 0$, we see that only five intermediate cases can occur: Either $W(x) = \pm 1/(ax+b) + ax + b$ admits ten (odd) symmetries; or $W(x) = c/(ax+b) + ax + b$ admits four (odd) symmetries if $c \neq \pm 1$; or $W(x) = W_0(x) + ax + b$ admits four (odd) symmetries if W_0 is a solution of Eq. (3.23); or

$$W(x) = \frac{d\exp(\pm ax^2)}{d \mp c\int\exp(\pm ax^2)dx} + ax$$

admits three (odd) symmetries; or

$$W(x) = \frac{c\exp(\frac{1}{2}b^2x^2 \mp 2bx)}{d \pm c\int\exp(\frac{1}{2}b^2x^2 \mp 2bx)dx} \mp \frac{1}{2}b^2x + b$$

admits three (odd) symmetries.

Consequently, we can summarize in Table 2 the specific families of superpotentials admitting odd supersymmetries.

Already mentioned in the even context, we notice once again that the free case, the linear case, and the harmonic oscillator case are also involved within the same class $1'$ and that their 12 odd supersymmetries can be quoted [in correspondence with (3.9)] in the forms

$$[(H_B, C_\pm, I, P_\pm)\sigma_1] \quad \text{and} \quad [(H_B, C_\pm, I, P_\pm)\sigma_2] \tag{3.26}$$

according to old notations [9] and remembering that σ_1 and σ_2 are the two odd matrices of Cl_2.

Table 2.

Classes	Number of $\mathcal{E}$ supersymmetries	Explicit forms of associated superpotentials	Characteristics
$1'$	12	$W(x) = ax + b$	Free case Linear case Harmonic oscillator [9]
$2'$	10	$W(x) = c/x$	$c = \pm 1$
$3'$	10	$W(x) = ax + b + c/(x + b/a)$	$a \neq 0,\ c = \pm 1$
$4'$	4	$W(x) = ax + b + c/(x + b/a)$	$a \neq 0,\ c \neq \pm 1$
$5'$		$W(x) = c/x$	$c \neq \pm 1$
$6'$	4	$W(x) = b + W_0(x)$ if $W_0''' - 6W_0^2 W' - 12bW_0W_0' - 6b^2W_0' = 0$	$a = 0$
$7'$		$W(x) = ax + b + W_0(x)$ if $W_0''' - 6W_0^2 W' - 12(ax+b)W_0W_0'$ $+ 12(ax+b)^2 W_0' + 36a(ax+b)W_0 = 0$	$a \neq 0$
$8'$	3	$W(x) = ax + \frac{c \exp(\pm ax^2)}{d \mp c \int \exp(\pm ax^2) dx}$	$a \neq 0,\ c \neq 0$
$9'$		$W(x) = \mp\frac{1}{2}b^2 x + b + \frac{c \exp(\frac{1}{2}b^2x^2 \mp 2bx)}{d \pm c \int \exp(\frac{1}{2}b^2x^2 \mp 2bx) dx}$	$b \neq 0,\ c \neq 0$
$10'$	2	$W(x) = W_0(x) + ax + b,\ W_0 \neq$ above forms	$\cdots$

3.3 Supersymmetries and invariance superalgebras in SSQM

We can now superpose the results on even and odd supersymmetries collected in Tables 1 and 2, respectively. If such a superposition is relatively direct for superpotentials belonging to the classes 1, 2, 3, 4 and $1'$, $2'$, $3'$, $4'$ due to their similar forms, it is evident that the superposition of all other information becomes relatively tedious and that it is not interesting to insist on the corresponding properties.

Besides the information on the maximal number 24 ($= 12 + 12$) of supersymmetries associated with the classes 1 and $1'$, as well as on the minimal number 6 ($= 4 + 2$) associated with the classes 10 and $10'$, we can distinguish what are the superstructures generated by some subsets of operators. We have determined that only three (closed) invariance Lie superalgebras can be pointed out. We have summarized these structures and their associated properties in Table 3. We evidently recover the expected results [9] for the supersymmetric harmonic oscillator but also find new ones for the Coulomb and the Calogero problems.

We notice that only a few cases are priviledged in this supersymmetric context when we require invariance su perstructures.

4 Comments and conclusions

As already noticed (and expected), the free case leads to the largest number (24) of Supersymmetries and to the largest (13-dimensional) kinematical invariance superalgebra. These results are also valid for the linear case and the harmonic oscillator context, which, both, are isomorphic to the free case as it is true in NRQM. In fact, let us point out that for those three cases we are dealing with the superpotential given by (see Table 3):

$$W(x) = ax + b, \tag{4.1}$$

so that the corresponding supersymmetric Hamiltonian (1.2) is

$$H_{SS} = \frac{1}{2}p^2 + \frac{1}{2}(a^2x^2 + 2abx + b^2) + \frac{1}{2}a\sigma_3. \tag{4.2}$$

Then, by using the unitary transformation

$$U = \exp[(i/2)at\sigma_3], \tag{4.3}$$

immediately gel for the operator $\Delta_{SS} \equiv$ (3.3), i.e.,

$$\Delta_{SS} \equiv -i\partial_t + H_{SS}, \tag{4.4}$$

that

$$U\Delta_{SS}U^{-1} = \left[-i\partial_t - \frac{1}{2}\partial_x^2 + \frac{1}{2}(a^2x^2 + 2abx + b^2)\right] I_2. \tag{4.5}$$

This expression corresponds to the NRQM context characterized by the potential (2.6), but amplified by the identity matrix I_2 belonging to the Clifford algebra Cl_2. We thus recover the six symmetries obtained by Niederer [1] multiplied here (four times) by the elements of Cl_2 leading to $12\mathcal{E} + 12\mathcal{O}$, i.e., to 24 supersymmetries [9] characterized in Eqs. (3.9) and (3.26). Among them only 13 close and lead to the largest kinematical invariance superalgebra osp(2/2)□sh(2/2) [14]. Associated with these comments are changes of variables easily determined from the formulas (2.9)–(2.11) and directly connected with other results [1, 15] when, for example, $a = \omega$ and $b = 0$.

Let us also insist on the specific interest of the simplest Witten superalgebra sqm(2) considered in Eqs. (3.17) and recovered as a part of the minimal closed superstructure found in Table 3. We immediately notice that

$$\mathrm{osp}(2/2) \supset \mathrm{osp}(2/1) \supset \mathrm{sqm}(2), \tag{4.6}$$

a physical chain of particular interest.

Besides the general conclusions that can be drawn from the tables and more particularly from Table 3, let us recall that different types of superpotentials have already been studied in SSQM (see more particularly the reviews of D'Hoker et al [16] and of Lahiri et al [17]). All these superpotentials fall into one of the classes 1–4 contained in Table 3, as well as those we have considered as partner potentials in parasupersymmetric quantum mechanics [18].

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Table 3.

	Superpotentials		Number of supersymmetries $\mathcal{E}$	$\mathcal{O}$	$N = \mathcal{E} + \mathcal{O}$	Invariance Lie superalgebras Dimension (d)	Superstructure	Characteristics
1	$W(x) = ax + b$		12	12	24	$d = 13$	osp(2/2)□sh(2/2)	Free case Linear case Harmonic oscillator [9]
2	$W(x) = c/x$	$c = \pm 1,$	10	10	20	$d = 7$	[osp(2/1)□so(2)] ⊕ gl(1)	Coulomb
		$c \neq \pm 1,\ c \neq 0$	8	4	12			Calogero
3	$W(x) = ax + c/x$	$c = \pm 1,$	10	10	20			
		$c \neq \pm 1,\ c \neq 0$	8	4	12			
4	$W(x) \neq$ above forms		$8, 6,$ or 4	$4, 3,$ or 2	$12 \leq N \leq 6$	$d = 5$	[sqm(2)□so(2)] ⊕ gl(1)	$\cdots$

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